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NOTE ON VARIATIONAL PROBLEMS OF TRANSONIC GAS FLOWS

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Behavior of the function $\nu(x)$ proportional to the variation of the stream function on the sonic line is studied. The conditions obtained in [1] for determination of parameters not defined in the statement of the variational problem are made more precise. It is shown that the parameter describing the displacement of the boundary of the varied flow relative to the basic flow, is not essential. The direct problem of a plane parallel transonic gas flow was reduced to a boundary value problem for a mixed second order linear differential equation in [2, 3], under the assumption that the flow did not differ appreciably from the specific flow.

The direct problem of transonic gas flow through a symmetric, plane parallel Laval nozzle with nearly rectilinear walls forming a certain angle and the problem of a sonic gas flow past a wedge-like profile with the zero angle of attack, were solved in [1]. The corresponding boundary value problems were reduced to the following singular integral equation for the function $\nu(x)$:

$$\nu(x)+\lambda\int_0^1 \nu(t)K(x,t)\,dt=g(x) \tag{1}$$

$$K(x,t)=\sum_{n=-\infty}^{+\infty}\left(\frac{x}{|2n+t|}\right)^{2\beta}\left(\frac{1}{2n+t+x}+\frac{1}{2n+t-x}\right)$$

$$\lambda=\frac{\cos\pi\beta}{\pi(\sin\pi\beta-1)}$$

The function $g(x)$ is expressed in terms of the boundary conditions of the problem. It is continuous on the interval $(0, 1]$ and can become infinite of order -2β at the point $x=0$. Consequently $f(x)=x^{-2\beta}g(x)$ is continuous and bounded on the interval $[0, 1]$. An approximate solution $\nu_1(x)$ of (1), taking into account the fact that $\nu(x)$ is singular at $x=0$, was obtained in [1]

$$\nu_1(x)=\cos^2\pi\mu\, x^{2\beta}[f(x)-\lambda I(x)],\qquad \mu={}^1/_2({}^1/_2+\beta) \tag{2}$$

$$I(x)=\frac{\pi}{2}\int_0^1\left(\frac{\operatorname{tg}{}^1/_2\pi t}{\operatorname{tg}{}^1/_2\pi x}\right)^{2\mu}\left[\operatorname{ctg}\frac{\pi}{2}(t-x)+\operatorname{ctg}\frac{\pi}{2}(t+x)\right]f(t)\,dt$$

Let us investigate the singular integral $I(x)$, passing to the new variables

$$y=\sin^2{}^1/_2\pi x,\qquad \tau=\sin^2{}^1/_2\pi t,\qquad f(x)=\varphi(y)$$

and changing the variable of integration

$$\xi=\frac{(1-\tau)y}{(1-y)\tau},\qquad \tau=\frac{y}{y+(1-y)\xi}$$

We have

$$I(x)=\int_0^\infty \frac{\varphi(\tau)\,\xi^{-\mu}d\xi}{1-\xi}+(1-y)\int_0^\infty \frac{\varphi(\tau)\,\xi^{-\mu}d\xi}{y+(1-y)\,\xi}$$

Returning in the last integral to the former variables we obtain

$$I(x)=\int_0^\infty \varphi\left(\frac{y}{y+(1-y)\,\xi}\right)\frac{\xi^{-\mu}\,d\xi}{1-\xi}+\pi_1\left(\operatorname{ctg}\frac{\pi}{2}x\right)^{2\mu}\int_0^1\left(\operatorname{tg}\frac{\pi}{2}t\right)^{\beta-1/2} f(t)\,dt \qquad (3)$$

Since $-1/2<\beta<0$ and $0<\mu<1/4$, both integrals in (3) exist; the first of them is regarded as representing its Cauchy principal value. The function $I(x)$ is continuous on the interval $(0, 1]$ and near the point $x=0$ the following expansion holds:

$$I(x)=A+O(x)+x^{-2\mu}\,[2^{2\mu}\pi^{1-2\mu}B+O(x)] \qquad (4)$$

$$A=f(0)\int_0^\infty \frac{\xi^{-\mu}\,d\xi}{1-\xi}, \qquad B=\int_0^1\left(\operatorname{tg}\frac{\pi}{2}t\right)^{\beta-1/2} f(t)\,dt$$

Having computed the singular integral (see [4], formula 3.222.2) and inserting the expansion (4) into (2), we obtain

$$v_1(x)=\cos^2\pi\mu\,\{x^{2\beta}\,[f(0)-f(0)+O(x)]-\lambda x^{\beta-1/2}\,[2^{2\mu}\pi^{1-2\mu}B+O(x)]\},$$

$$\frac{1}{\lambda}=\pi\operatorname{ctg}[\pi(1-\mu)]$$

Consequently, when $x\to 0$, the function $v_1(x)$ became infinity of order $1/2-\beta$. The necessary and sufficient condition for $v_1(x)$ to be bounded is (see [1], formula (5.21))

$$B=\int_0^1\left(\operatorname{tg}\frac{\pi}{2}t\right)^{\beta-1/2} t^{-2\beta}\,g(t)\,dt=0 \qquad (5)$$

It is not necessary to require that $g(x)$ should be bounded, i.e. that $f(0)=0$ (see [1], formula (5.20)). Condition (5) is used for determination of the variation of gas consumption in the direct problem of the Laval nozzle, or the coefficient accompanying the singular solution in the problem of transonic flow past a wedge-like profile.

The relation $f(0)=0$ was used in [1] to determine the shift of the boundary of the varied flow relative to the basic flow. Since this condition is not necessary, the shift must be determined by calculating minimum distances between the boundaries.

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